

Planar graphs and Stanley's Chromatic Functions

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Abstract

This article is dedicated to the study of positivity phenomena for the chromatic symmetric function of a graph with respect to various bases of symmetric functions.

We give a new proof of Gasharov's theorem on the Schur-positivity of the chromatic symmetric function of a $(3 + 1)$ -free poset. We present a combinatorial interpretation of the Schur-coefficients in terms of planar networks. Compared to Gasharov's proof, it gives a clearer visual illustration of the cancellation procedures and is quite similar in spirit to the proof of monomial positivity of Schur functions via the Lindström–Gessel–Viennot lemma.

We apply a similar device to the e -positivity problem of chromatic functions. Following Stanley, we analyze certain analogs of symmetric functions attached to graphs instead of working with chromatic symmetric functions of graphs directly. We introduce a new combinatorial object: the *correct* sequences of unit interval orders, and, using these, we reprove monomial positivity of G -analogues of the power sum symmetric functions.

1 Introduction

Let G be a finite graph, $V(G)$ - the set of vertices of G , $E(G)$ - the set of edges of G .

Definition 1.1. A *proper coloring* c of G is a map

$$c : V \rightarrow \mathbb{N}$$

such that no two adjacent vertices are colored in the same color.

For each coloring c we define a monomial

$$x^c = \prod_{v \in V} x_{c(v)},$$

where $x_1, x_2, \dots, x_n, \dots$ are commuting variables. We denote by $\Pi(G)$ the set of all proper colorings of G , and by Λ the ring of symmetric functions in the infinite set of variables $\{x_1, x_2, \dots\}$.

In [2], Stanley defined the chromatic symmetric function of a graph.

Definition 1.2. The *chromatic symmetric function* $X_G \in \Lambda$ of a graph G is the sum of the monomials x^c over all proper colorings of G :

$$X_G = \sum_{c \in \Pi(G)} x^c.$$

Definition 1.3. Denote by e_m the m -th elementary symmetric function:

$$e_m = \sum_{i_1 < i_2 < \dots < i_m} x_{i_1} \cdot x_{i_2} \cdot \dots \cdot x_{i_m},$$

where $i_1, \dots, i_k \in \mathbb{N}$. Given a non-increasing sequence of positive integers (we will call these *partitions*)

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k), \lambda_i \in \mathbb{N},$$

we define the elementary symmetric function $e_\lambda = \prod_{i=1}^k e_{\lambda_i}$. These functions form a basis of Λ .

For a natural number k , we denote by 1^k the partition λ of length k , where

$$\lambda_1 = \lambda_2 = \dots = \lambda_k = 1.$$

Definition 1.4. A symmetric function $X \in \Lambda$ is *e-positive* if it has non-negative coefficients in the basis of the elementary symmetric functions.

Definition 1.5. Denote by p_m the m -th power sum symmetric function:

$$p_m = \sum_{i \in \mathbb{N}} x_i^m.$$

Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k)$, we define the power sum symmetric function $p_\lambda = \prod_{i=1}^k p_{\lambda_i}$. These functions also form a basis of Λ .

Definition 1.6. Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k)$, we define the monomial symmetric function

$$m_\lambda = \sum_{i_1 < i_2 < \dots < i_k} \sum_{\lambda' \in S_k(\lambda)} x_{i_1}^{\lambda'_1} \cdot x_{i_2}^{\lambda'_2} \cdot \dots \cdot x_{i_k}^{\lambda'_k},$$

where the inner sum is taken over the set of all permutations of the sequence λ , denoted by $S_k(\lambda)$.

Example 1.7. The chromatic symmetric function of K_n , the complete graph on n vertices, is *e-positive*: $X_{K_n} = n! e_n$.

Definition 1.8. For a poset P , the *incomparability graph*, $\text{inc}(P)$, is the graph with elements of P as vertices, where two vertices are connected if and only if they are not comparable in P .

Definition 1.9. Given a pair of natural numbers $a, b \in \mathbb{N}^2$, we say that a poset P is $(a+b)$ -free if it does not contain a length- a and a length- b chain, whose elements are mutually incomparable.

Definition 1.10. A unit interval order (UIO) is a partially ordered set which is isomorphic to a finite subset of $U \subset \mathbb{R}$ with the following poset structure:

$$\text{for } u, w \in U : u \succ w \text{ iff } u \geq w + 1.$$

Thus u and w are incomparable precisely when $|u - w| < 1$ and we will use the notation $u \sim w$ in this case.

Theorem 1.11 (Scott-Suppes [1]). *A finite poset P is a UIO if and only if it is $(2+2)$ - and $(3+1)$ -free.*

Stanley [2] initiated the study of incomparability graphs of $(3+1)$ -free partially ordered sets. Analyzing the chromatic symmetric functions of these incomparability graphs, Stanley [2] stated the following positivity conjecture.

Conjecture 1.12 (Stanley). *If P is a $(3+1)$ -free poset, then $X_{\text{inc}(P)}$ is *e-positive*.*

For a graph G let us denote by $c_\lambda(G)$ the coefficients of X_G with respect to the *e*-basis. We omit the index G whenever this causes no confusion:

$$X_G = \sum_{\lambda} c_\lambda e_\lambda.$$

Conjecture 1.12 has been verified with the help of computers for up to 20-element posets [6]. In 2013, Guay-Paquet [6] showed that to prove this conjecture, it would be sufficient to verify it for the case of $(3+1)$ - and $(2+2)$ -free posets, i.e. for unit interval orders (see Theorem 1.11). More precisely:

Theorem 1.13 (Guay-Paquet). *Let P be a $(3+1)$ -free poset. Then, $X_{\text{inc}(P)}$ is a convex combination of the chromatic symmetric functions*

$$\{X_{\text{inc}(P')} \mid P' \text{ is a } (3+1)\text{- and } (2+2)\text{-free poset}\}.$$

The strongest general result in this direction is that of Gasharov [3].

Definition 1.14. For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k)$, define the Schur functions $s_\lambda = \det(e_{\lambda_i^* + j - i})_{i,j}$, where λ^* is the conjugate partition to λ . The functions $\{s_\lambda\}$ form a basis of Λ .

Definition 1.15. A symmetric polynomial X is *s-positive* if it has non-negative coefficients in the basis of Schur functions.

Obviously, a product of *e*-positive functions is *e*-positive. This also holds for *s*-positive functions. Thus, the equality $e_n = s_{1^n}$ implies that *e*-positive functions are *s*-positive, and thus *s*-positivity is weaker than *e*-positivity.

Theorem 1.16 (Gasharov). *If P is a $(3+1)$ -free poset, then $X_{\text{inc}(P)}$ is *s-positive*.*

Gasharov proved *s*-positivity by constructing so-called *P*-tableau and finding a one-to-one correspondence between these tableau and *s*-coefficients [3]. However, *e*-positivity conjecture 1.4 is still open. The strongest known result on the *e*-coefficients was obtained by Stanley in [2]. He showed that sums of *e*-coefficients over the partitions of fixed length are non-negative:

Theorem 1.17 (Stanley). *For a finite graph G and $j \in \mathbb{N}$, suppose*

$$X_G = \sum_{\lambda} c_{\lambda} e_{\lambda},$$

and let $\text{sink}(G, j)$ be the number of acyclic orientation of G with j sinks. Then

$$\text{sink}(G, j) = \sum_{l(\lambda)=j} c_{\lambda}.$$

Remark 1.18. *By taking $j = 1$, it follows from the theorem that c_n is non-negative.*

Stanley in [2] showed that for $n \in \mathbb{N}$ and the unit interval order $P_n = \{\frac{i}{2}\}_{i=1}^n$, the corresponding $X_{\text{inc}(P_n)}$ is *e*-positive, while *e*-positivity for the UIOs

$$P_{n,k} = \left\{ \frac{i}{k+1} \right\}_{i=1}^n$$

with $k > 1$ has not yet been proven. It was checked for small n and some k (see [2]).

In this article, we give a new proof of Gasharov's theorem, which presents a combinatorial interpretation of the *s*-coefficients in terms of planar networks. Compared to Gasharov's proof, it gives a clearer visual illustration of the cancellation procedures and resembles the proof of monomial positivity of Schur functions using Lindström–Gessel–Viennot Lemma [9]. This allows us to look at the positivity problematics from a slightly different perspective: instead of working with the chromatic symmetric function of a graph directly, we analyze families of *G*-symmetric functions, described in Section 2, first time proposed by Stanley in [7].

Next, we introduce *correct sequences* (abbreviated as *corrects*), defined below. These play a major role in the article.

Definition 1.19. Let U be a UIO. We will call a sequence $\vec{w} = (w_1, \dots, w_k)$ of elements of U *correct* if

- $w_i \not\sim w_{i+1}$ for $i = 1, 2, \dots, k-1$
- and for each $j = 2, \dots, k$, there exists $i < j$ such that $w_i \not\sim w_j$.

Every sequence of length 1 is correct, and sequence (w_1, w_2) is correct precisely when $w_1 \sim w_2$. The second condition (supposing that the first one holds) may be reformulated as follows: for each $j = 1, \dots, k$, the subset $\{w_1, \dots, w_j\} \subset U$ is connected with respect to the graph structure (U, \sim) . Using this notation, we prove the following theorems.

Theorem 1.20. *Let $X_{\text{inc}(U)} = \sum_{\lambda} c_{\lambda} e_{\lambda}$ be a chromatic symmetric function of the n -element unit interval order U . Then c_n is equal to the number of corrects of length n , in which every element of U is used exactly once.*

Corollary 1.20.1. *Let $X_{\text{inc}(P)} = \sum_{\lambda} c_{\lambda} e_{\lambda}$ be a chromatic symmetric function of n -element $(3+1)$ -free poset P , then c_n is a nonnegative integer.*

Indeed, positivity for the general case follows from Theorem 1.13, which presents the chromatic symmetric function of a $(3 + 1)$ -free poset as a convex combination of the chromatic symmetric functions of unit interval orders.

Stanley [7] and Chow [5] showed the positivity of c_n for $(3 + 1)$ -free posets using combinatorial techniques, and linked e -coefficients with the acyclic orientations of the incomparability graphs. Nevertheless, their proofs do not give visual interpretation of the cancellation procedures.

The article is structured as follows: in Section 2, we describe the G -homomorphism introduced by Stanley in [7], which is essential for our approach. The new proof of Gasharov's theorem (Theorem 1.16) is presented in Section 3.1. The proof of Theorem 1.20 and positivity of G -power sum symmetric functions is can be found in Section 3.2.

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2 Stanley's G -homomorphism

For a graph G , Stanley [7, p. 6] defined G -analogues of the standard families of symmetric functions. Let G be a finite graph with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G)$. We will think of the elements of $V(G)$ as commuting variables.

Definition 2.1. For a positive integer i , $1 \leq i, \leq n$, we define the G -analogues of the elementary symmetric polynomials, or *the elementary G -symmetric polynomials*, as follows

$$e_i^G = \sum_{\substack{\#S=i \\ S\text{-stable}}} \prod_{v \in S} v,$$

where the sum is taken over all i -element subsets S of V , in which no two vertices form an edge, i.e. stable subsets. We set $e_0^G = 1$, and $e_i^G = 0$ for $i < 0$.

Note that these polynomials are not necessarily symmetric.

Let $\Lambda_G \subset \mathbb{R}[v_1, \dots, v_n]$ be the subring generated by $\{e_i^G\}_{i=1}^n$. The map $e_i \mapsto e_i^G$ extends to a ring homomorphism $\phi_G : \Lambda \rightarrow \Lambda_G$, called the *G -homomorphism*. For $f \in \Lambda$, we will use the notation f^G for $\phi_G(f)$.

Example 2.2. Given a partition $\lambda = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$, $k \in \mathbb{N}$, we have

$$e_\lambda^G = \prod_{i=1}^k e_i^G,$$

$$s_\lambda^G = \det(e_{\lambda_i^* + j - i}^G).$$

For an integer function $\alpha : V \rightarrow \mathbb{N}$ and $f^G \in \Lambda_G$, let

$$v^\alpha = \prod_{v \in V} v^{\alpha(v)},$$

and $[v^\alpha]f^G$ stands for the coefficient of v^α in the polynomial $f^G \in \Lambda_G$.

Let G^α denote the graph, obtained by replacing every vertex v of G by the complete subgraph of size $\alpha(v)$: $K_{\alpha(v)}^v$. Given vertices u and v of G , a vertex of $K_{\alpha(v)}^v$ is connected to a vertex of $K_{\alpha(u)}^u$ if and only if u and v form an edge in G .

Considering the Cauchy product [8, ch. 4.2], Stanley [7, p. 6] found a connection between the G -analogues of symmetric functions and X_G . Following Stanley [7], we set

$$T(x, v) = \sum_{\lambda} m_{\lambda}(x) e_{\lambda}^G(v),$$

where the sum is taken over all partitions. Then

$$[v^\alpha]T(x, v) \prod_{v \in V} \alpha(v)! = X_{G^\alpha}. \quad (1)$$

Using the Cauchy identity

$$\sum_{\lambda} s_{\lambda}(x) s_{\lambda^*}(y) = \sum_{\lambda} m_{\lambda}(x) e_{\lambda}(y) = \sum_{\lambda} e_{\lambda}(x) m_{\lambda}(y)$$

and applying the G -homomorphism, one obtains:

$$T(x, v) = \sum_{\lambda} m_{\lambda}(x) e_{\lambda}^G(v) = \sum_{\lambda} s_{\lambda}(x) s_{\lambda^*}^G(v) = T(v, x) = \sum_{\lambda} e_{\lambda}(x) m_{\lambda}^G(v). \quad (2)$$

An immediate consequence of the formulas (1) and (2) is the following result of Stanley:

Theorem 2.3 (Stanley). *For every finite graph G*

1. $X_{G^{\alpha}}$ is s -positive for every $\alpha : V(G) \rightarrow \mathbb{N}$ if and only if $s_{\lambda}^G \in \mathbb{N}[V(G)]$ for every partition λ .
2. $X_{G^{\alpha}}$ is e -positive for every $\alpha : V(G) \rightarrow \mathbb{N}$ if and only if $m_{\lambda}^G \in \mathbb{N}[V(G)]$ for every partition λ .

Remark 2.4. If $X_{G^{\alpha}} = \sum_{\lambda} c_{\lambda}^{\alpha} e_{\lambda}$, then $c_{\lambda}^{\alpha} = [v^{\alpha}] m_{\lambda}^G$. Hence, monomial positivity of m_{λ}^G is equivalent to the positivity of c_{λ}^{α} for every α .

The proofs of positivity of G -power sum symmetric functions and Schur G -symmetric functions for the case of unit interval orders can be found in [10].

3 Proofs of the theorems

It follows from Theorem 2.3 that to prove that the graph G is s -positive, it is enough to show the monomial positivity of its G -Schur polynomials. On the other hand, Guay-Paquet (Theorem 1.13) showed that it is sufficient to check s -positivity for unit interval orders in order to prove it for the general case of $(3 + 1)$ -free posets. Therefore, in the following paragraph 3.1, we analyze the functions s_{λ}^G for the case $G = \text{inc}(U)$, where U is UIO.

3.1 A new proof of Gasharov's theorem

Given unit interval order U , we arrange the elements of U according to their order on the real line. For instance, the incomparability graph of $U_8 = \{v_i = \frac{i}{2}\}_{i=1}^8$, the 1-chain graph with 8 vertices, has the following labeling:



Figure 1: The incomparability graph of U_8 .

A key tool in our work is the Lindström–Gessel–Viennot lemma [9]. Let Γ be a finite directed acyclic (i.e. without directed cycles) graph with set of vertices $V(\Gamma)$ and set of edges $E(\Gamma)$. Let $w : E(\Gamma) \rightarrow R$ be a weighting of the edges with values in some commutative ring R . For every directed path ρ , denote by $w(\rho)$ the product of the weights of the edges in the path. Then, for every two vertices a and b of Γ , let

$$e(a, b) = \sum_{\rho: a \rightarrow b} w(\rho),$$

where the sum is taken over all paths from a to b .

Definition 3.1. Let $n \in \mathbb{N}$, and let us fix two ordered n -element subsets

$$A = (a_1, a_2, \dots, a_n) \subset V(\Gamma) \text{ and } B = (b_1, b_2, \dots, b_n) \subset V(\Gamma),$$

called *base* and *destination* vertices, correspondingly. We will call a collection $\vec{\rho} = (\rho_1, \dots, \rho_n)$ of paths in Γ a *multipath* from A to B if there is a permutation σ on $\{1, 2, \dots, n\}$ such that $\rho_i : a_i \rightarrow b_{\sigma(i)}$, $i = 1, \dots, n$. Given a multipath $\vec{\rho}$, we denote the permutation σ by $\sigma_{\vec{\rho}}$. We call $\vec{\rho}$ *non-intersecting*, if $\rho_i \cap \rho_j = \emptyset$ for $i \neq j$.

Theorem 3.2 (Lindström–Gessel–Viennot). *Let Γ , $w : E(\Gamma) \rightarrow R$ be a weighted locally finite acyclic graph as above, $n \in \mathbb{N}$, $A = (a_1, a_2, \dots, a_n) \subset V(\Gamma)$ and $B = (b_1, b_2, \dots, b_n) \subset V(\Gamma)$. Define the matrix*

$$M_{A,B} = \begin{pmatrix} e(a_1, b_1) & e(a_1, b_2) & e(a_1, b_3) & \dots & e(a_1, b_n) \\ e(a_2, b_1) & e(a_2, b_2) & e(a_2, b_3) & \dots & e(a_2, b_n) \\ \dots & \dots & \dots & \dots & \dots \\ e(a_n, b_1) & e(a_n, b_2) & e(a_n, b_3) & \dots & e(a_n, b_n) \end{pmatrix}$$

Then, the following equality holds in the ring R :

$$\det(M_{A,B}) = \sum_{\substack{\vec{\rho}: A \rightarrow B \\ \text{non-int.}}} \text{sign}(\sigma_{\vec{\rho}}) \cdot \prod_{i=1}^n w(\rho_i),$$

where the sum is taken over all non-intersecting multipaths.

Remark 3.3. *It follows from Theorem 1.13 that to prove Gasharov's theorem, it is sufficient to verify it for unit interval orders. Here we prove Gasharov's theorem for this case.*

Theorem 3.4. *Let (U, \succ) be a unit interval order, $G = \text{inc}(U)$ its incomparability graph. Then, for every partition λ , $s_\lambda^G \in \mathbb{N}[V(G)]$.*

Proof. We prove the monomial positivity of s_λ^G by constructing a special directed graph Γ_G , the grid of G , and applying the Lindström–Gessel–Viennot theorem to Γ_G .

The vertices of Γ_G are given by the pairs (i, j) , where $i \in 1, \dots, n+1$ and $j \in \mathbb{N}$. Then, for every $i \in \{1, \dots, n\}$, we denote by $\text{next}(i) = \min \{j \mid v_j \succ v_i\}$. If such v_j does not exist, then we define $\text{next}(i) = n+1$. From every vertex (i, j) , $j < n+1$, we draw a directed edge to the vertex $(i, j+1)$ with the weight 1, and a directed edge to $(i+1, \text{next}(j))$ with the weight v_i . Note that Γ_G is planar if U is a unit interval order.

For instance, for the graph U_8 , mentioned above, the grid Γ_{U_8} is as follows:

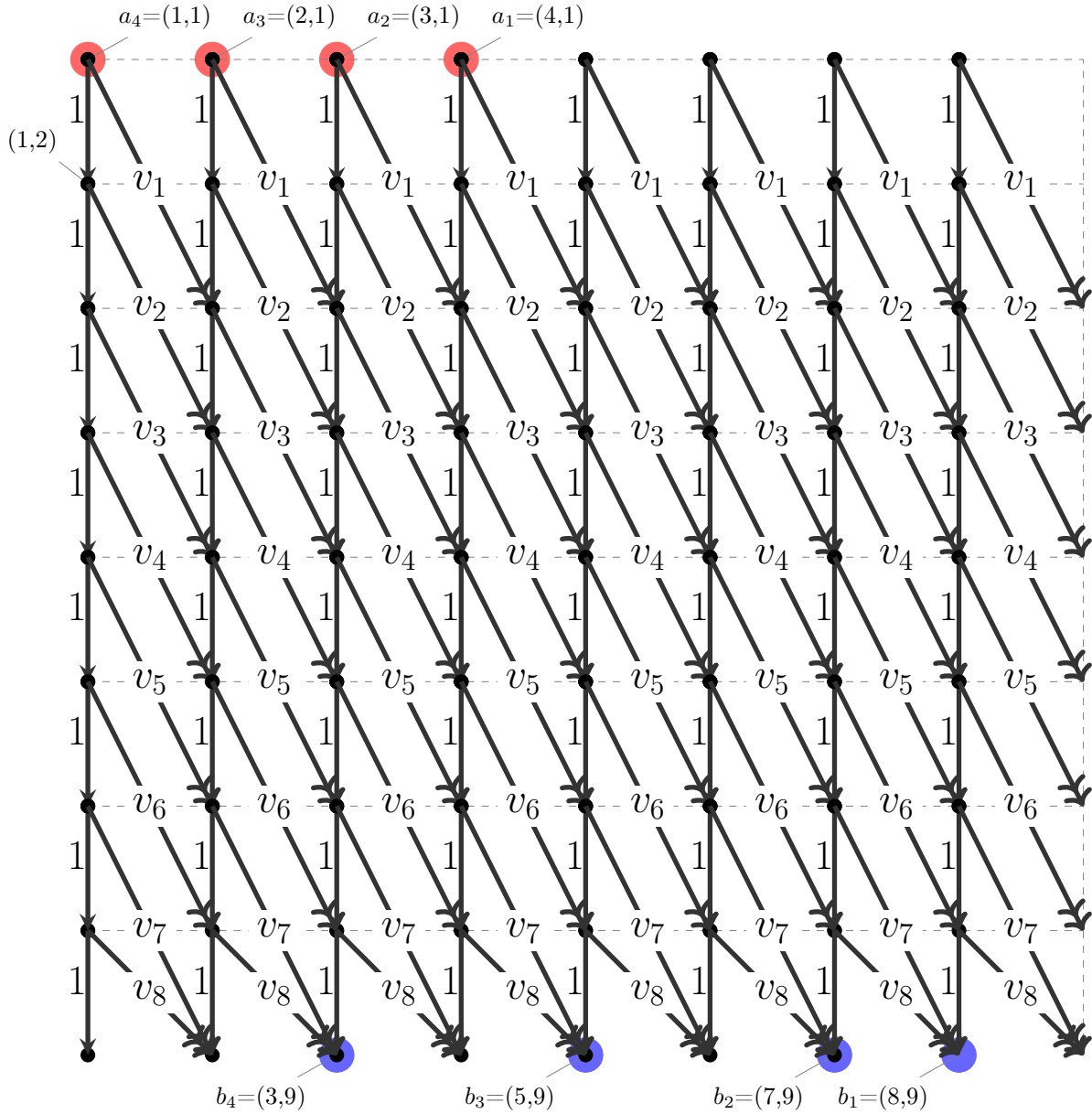


Figure 2: The grid Γ_{U_8} .

Here, the base vertices A , $(1, 1)$, $(2, 1)$, $(3, 1)$, and $(4, 1)$, are on the top, and are colored in red. The destination vertices B , $(3, 9)$, $(5, 9)$, $(7, 9)$, and $(8, 9)$, at the bottom, and are colored in blue.

It easily follows from the definition of Γ_G that, for positive integers i and j , we have

$$e((i, 1), (i + j, n + 1)) = e_j^G.$$

Note that we use the notation $e(a, b)$ for the sum over all weights of paths from vertex a to vertex b , and we use a similar notation e_λ^G for the G -elementary symmetric functions. This is not a coincidence: for the graphs we will consider in this article and to which we apply Theorem 3.2, $e(a, b)$ will turn out to be the elementary G -symmetric function.

Now, let $k \in \mathbb{N}$, and fix a partition $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$. Let

$$A = \{a_1 = (k, 1), a_2 = (k - 1, 1), \dots, a_i = (k + 1 - i, 1), \dots, a_k = (1, 1)\},$$

and

$$B = \{b_1 = (k + \lambda_1, n + 1), b_2 = (k - 1 + \lambda_2, n + 1), \dots, b_i = (k + 1 - i + \lambda_i, n + 1), \dots, b_k = (\lambda_k + 1, n + 1)\}.$$

Then we have

$$e(a_i, b_j) = e_{\lambda_i + j - i}^G.$$

The example of $\lambda = (4, 4, 3, 2)$ is shown on Figure 2.

Next, applying Theorem 3.2, we obtain

$$\det(e_{\lambda_i+j-i}^G) = \det(e(a_i, b_j)) = \sum_{\vec{\rho}: A \rightarrow B} \text{sign}(\sigma_{\vec{\rho}}) \prod_{i=1}^n w(\rho_i), \quad (3)$$

where the sum is taken over all non-intersecting multipaths from A to B . The permutation σ must be the identity permutation for all possible non-intersecting multipaths $\vec{\rho}$ since the grid Γ_G is planar. Thus, by the definition of s_{λ}^G , we have

$$s_{\lambda}^G = \det(e_{\lambda_i+j-i}^G) = \sum_{\substack{\vec{\rho}: A \rightarrow B \\ \text{non-intersecting}}} \prod_{i=1}^n w(\rho_i), \quad (4)$$

where the sum is taken over all non-intersecting multipaths from A to B . This proves the monomial positivity of s_{λ}^G . \square

3.2 Monomial positivity of the G -power sum functions.

Let us repeat the definition of a central notion for our work, that of *correct* sequences of elements of a unit interval order.

Definition 3.5. Let $(U, <)$ be a unit interval order, and $G = \text{inc}(U)$. We will call a sequence $\vec{w} = (w_1, \dots, w_k)$ of elements of U *correct* if

- $w_i \not\prec w_{i+1}$ for $i = 1, 2, \dots, k-1$
- and for each $j = 2, \dots, k$, there exists $i < j$ such that $w_i \not\prec w_j$.

We denote by P_k^U the set of all correct sequences (abbreviated as *corrects*) of length k . Since G is uniquely defined by U , and we are working only with UIO, here and below we use the U -index instead of G . The U -analogues of symmetric functions will be analyzed.

Theorem 3.6. Let U be a unit interval order and p_k^U the Stanley power-sum function of the corresponding incomparability graph. Then, for every natural k , we have

$$p_k^U = \sum_{\vec{w} \in P_k^U} w_1 \cdot \dots \cdot w_k \in N[U],$$

where the sum is taken over all corrects of length k .

Proof. To prove this theorem we express the power sum U -symmetric function p_k^U in terms of the elementary U -symmetric polynomials using the determinant formula:

$$p_k^U = \det \begin{vmatrix} e_1^U & 1 & 0 & \dots & \\ 2e_2^U & e_1^U & 1 & 0 & \dots \\ 3e_3^U & e_2^U & e_1^U & 1 & \\ \vdots & & & \ddots & \ddots \\ ke_k^U & e_{k-1}^U & \dots & & e_1^U \end{vmatrix}. \quad (5)$$

Note that this determinant is similar to the expression for $s_{(1^k)^*}^U$ in terms of the e -basis; only the first column is different:

$$s_{(1^k)^*}^U = \det \begin{vmatrix} e_1^U & 1 & 0 & \dots & \\ e_2^U & e_1^U & 1 & 0 & \dots \\ e_3^U & e_2^U & e_1^U & 1 & \\ \vdots & & & \ddots & \ddots \\ e_k^U & e_{k-1}^U & \dots & & e_1^U \end{vmatrix}. \quad (6)$$

Next, we take a partition $\lambda = 1^k$, a grid Γ_U , and vertices (see Theorem 3.2 and its use in Section 3.1)

$$A = \{a_1 = (k, 1), a_2 = (k-1, 1), \dots, a_i = (k+1-i, 1), \dots, a_k = (1, 1)\},$$

and

$$B = \{b_1 = (k+1, n+1), b_2 = (k, n+1), \dots, b_i = (k+1-i+1, n+1), \dots, b_k = (2, n+1)\},$$

corresponding to the partition λ on the grid. For instance, the grid for U_5 is as follows:

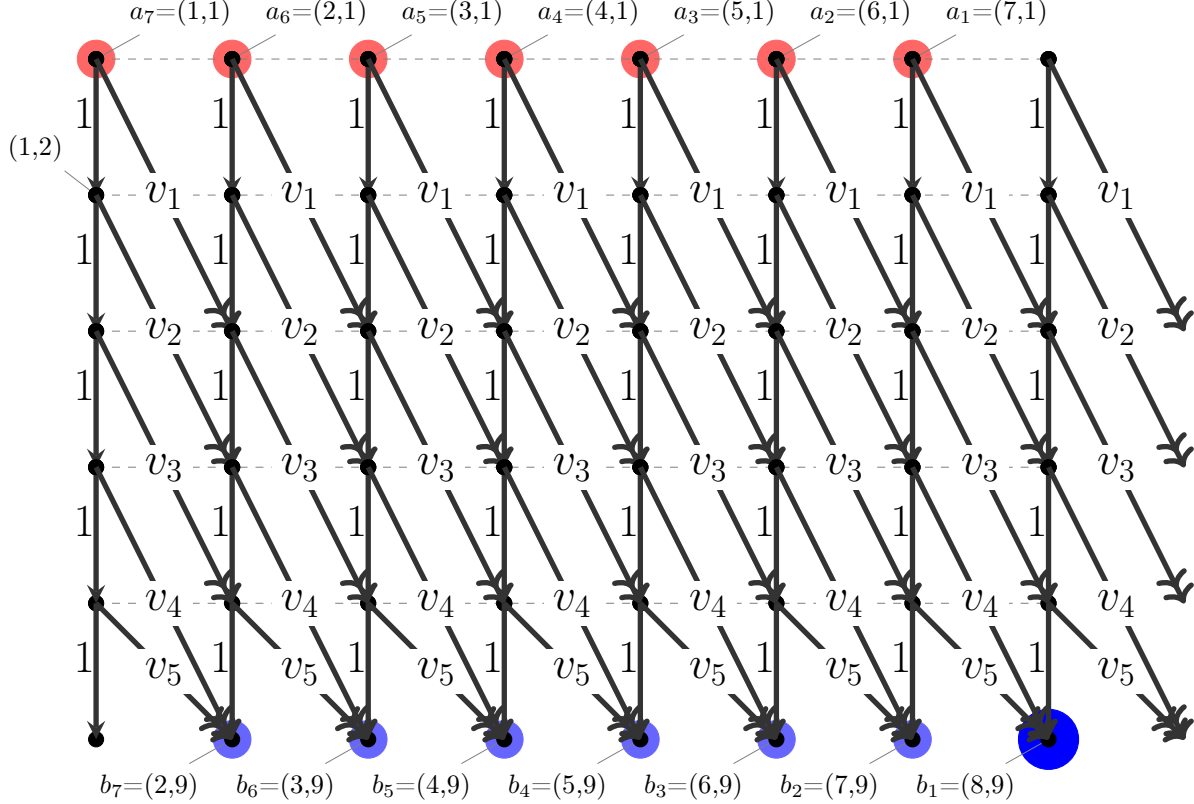


Figure 3: The grid Γ_{U_5} and vertices located with respect to partition $\lambda = 1^5$.

As in the section 3.1, we have

$$e(a_i, b_j) = e_{1+j-i}^U.$$

Recall that (see Theorem 3.2 for more details) for every directed path ρ on Γ_U , $w(\rho)$ denotes the product of the weights of the edges in the path. Denote by $w(\vec{\rho}) = (w(\rho_1), \dots, w(\rho_k))$ the vector of weight products over the paths of $\vec{\rho}$.

$$s_k^U = \det(e_{1+j-i}^U) = \det(e(a_i, b_j)) = \quad (7)$$

$$= \sum_{\vec{\rho}=(\rho_1, \dots, \rho_k): A \rightarrow B} \text{sign}(\sigma_{\vec{\rho}}) \prod_{i=1}^k w(\rho_i) = \sum_{\substack{(\rho_1, \dots, \rho_k): A \rightarrow B \\ \text{non-intersecting}}} \prod_{i=1}^k w(\rho_i). \quad (8)$$

To obtain p_k^U , we adjust the first column, multiplying every element by the number of its row:

$$p_k^U = \det \begin{vmatrix} e_1^U & 1 & 0 & \cdots & \\ 2e_2^U & e_1^U & 1 & 0 & \cdots \\ 3e_3^U & e_2^U & e_1^U & 1 & \\ \vdots & & & \ddots & \ddots \\ ke_k^U & e_{k-1}^U & \cdots & & e_1^U \end{vmatrix} = \det \begin{vmatrix} e(a_1, b_1) & 1 & 0 & \cdots & \\ 2e(a_2, b_1) & e(a_2, b_2) & 1 & 0 & \cdots \\ 3e(a_3, b_1) & e(a_3, b_2) & e(a_3, b_3) & 1 & \\ \vdots & & & \ddots & \ddots \\ ke(a_k, b_1) & e(a_k, b_2) & \cdots & & e(a_k, b_k) \end{vmatrix} = \quad (9)$$

$$= \sum_{\vec{\rho}=(\rho_1, \dots, \rho_k): A \rightarrow B} \text{sign}(\sigma_{\vec{\rho}}) \cdot \sigma_{\vec{\rho}}^{-1}(1) \prod_{i=1}^k w(\rho_i). \quad (10)$$

In this sum every multipath has a multiplier equal to the index of the vertex from A , from which the corresponding path goes to b_1 . We mark the vertex b_1 with a larger dot on the grid (Picture 3) to emphasize this. We cannot apply Theorem 3.2 here, as we did for s^U functions, to obtain positive sum.

We will use the following notations:

- If we have a path ρ on Γ_U , which goes from a to b through z , then let us denote by $\rho|_z$ the part of ρ from a to z , and by $\rho|_z -$ the part of ρ from z to b .
- If the end of the path ρ coincides with the starting point π , then we will write $\rho * \pi$ for the concatenation of the two paths
- For a pair of paths (ρ, π) , crossing in point z , we define the usual switch operation

$$\text{switch}_z(\rho, \pi) = (\rho|_z * \pi|_z, \pi|_z * \rho|_z).$$

- Given a multipath $\vec{\rho}$ with its paths ρ and π intersecting in point z , we define a multipath $\delta_z(\vec{\rho})$ by replacing (ρ, π) by $\text{switch}_z(\rho, \pi)$ in $\vec{\rho}$. Note that our map is defined correctly, because here we consider only multipaths for the partition $\lambda = 1^k$: it is obvious that 3 paths of $\vec{\rho}$ cannot intersect in one point. Note that

$$\text{sign}(\sigma(\vec{\rho})) = -\text{sign}(\sigma(\delta_z(\vec{\rho}))).$$

- Given an intersecting multipath $\vec{\rho}$, we denote by $z(\vec{\rho})$ (or just z , if it is clear which multipath is considered) its intersection point with minimum absciss and maximum ordinate, i.e. the leftmost lowest intersection point.

Next, we classify the set of multipaths in order to simplify the sum (10). Every path ρ can be uniquely defined by its weight, $w(\rho)$, which is a product over an increasing sequence (with respect to the relation \succ) of elements of U . Here, it is important to mention that incomparable elements of U can not be present in a weight of any path. Hence, every multipath $\vec{\rho} = (\rho_1, \dots, \rho_k)$ is in one to one correspondence with its weight vector $w(\vec{\rho}) = (w(\rho_1), \dots, w(\rho_k))$. Below, we will use the bar notation for the sets of multipaths. The corresponding sets of weight vectors will be defined using the same letters without bars.

- Let $\overline{\Omega}_k$ be the set of all multipaths $\vec{\rho} = (\rho_1, \dots, \rho_k)$ from A to B .
- Let \overline{I}_k be the set of all intersecting multipaths $\vec{\rho} \in \overline{\Omega}_k$, such that the two paths from $\vec{\rho}$, crossing at $z(\vec{\rho})$ do not end at b_1 .
- We denote by \overline{P}_k the set of multipaths $\vec{\rho} \in \overline{\Omega}_k$, such that $w(\vec{\rho})$ is correct:

$$\overline{P}_k = \{\vec{\rho} \in \overline{\Omega}_k \mid w(\vec{\rho}) \in P_k^U\}.$$

Note that if $\vec{\rho} \in \overline{P}_k$, then $\vec{\rho}$ is a non-intersecting multipath, since by the definition of correct $w(\vec{\rho})$ must be a tuple with non-decreasing elements (weights) with respect to the relation \prec . Hence,

$$\overline{I}_k \cap \overline{P}_k = \emptyset.$$

As a consequence, the sum (10) can be rewritten as

$$\sum_{\vec{\rho} \in \overline{\Omega}_k} \text{sign}(\sigma_{\vec{\rho}}) \cdot \sigma_{\vec{\rho}}^{-1}(1) \prod_{i=1}^k w(\rho_i) = \sum_{\vec{\rho} \in \overline{P}_k} \text{sign}(\sigma_{\vec{\rho}}) \cdot \sigma_{\vec{\rho}}^{-1}(1) \prod_{i=1}^k w(\rho_i) \quad (11)$$

$$+ \sum_{\vec{\rho} \in \overline{I}_k} \text{sign}(\sigma_{\vec{\rho}}) \cdot \sigma_{\vec{\rho}}^{-1}(1) \prod_{i=1}^k w(\rho_i) \quad (12)$$

$$+ \sum_{\vec{\rho} \in (\overline{\Omega}_k \setminus \overline{I}_k) \setminus \overline{P}_k} \text{sign}(\sigma_{\vec{\rho}}) \cdot \sigma_{\vec{\rho}}^{-1}(1) \prod_{i=1}^k w(\rho_i). \quad (13)$$

Let $\vec{\rho} \in \overline{I}_k$, then it is easy to see that $\delta_z(\vec{\rho}) \in \overline{I}_k$, and $\delta_z(\delta_z(\vec{\rho})) = \vec{\rho}$. Hence, δ_z is a sign-reversing involution on \overline{I}_k :

$$\text{sign}(\sigma(\vec{\rho})) = -\text{sign}(\sigma(\delta_z(\vec{\rho}))) \text{ and } \delta_z(\overline{I}_k) = \overline{I}_k$$

On the other hand, δ_z does not change the multiplier:

$$\sigma^{-1}(\delta_z(\vec{\rho}))(1) = \sigma_{\vec{\rho}}^{-1}(1).$$

Hence, the term (12) vanishes, and we have:

$$\sum_{\vec{\rho} \in \overline{I}_k} \text{sign}(\sigma_{\vec{\rho}}) \cdot \sigma_{\vec{\rho}}^{-1}(1) \prod_{i=1}^k w(\rho_i) = \delta_z \left(\sum_{\vec{\rho} \in \overline{I}_k} \text{sign}(\sigma_{\vec{\rho}}) \cdot \sigma_{\vec{\rho}}^{-1}(1) \prod_{i=1}^k w(\rho_i) \right) = - \sum_{\vec{\rho} \in \overline{I}_k} \text{sign}(\sigma_{\vec{\rho}}) \cdot \sigma_{\vec{\rho}}^{-1}(1) \prod_{i=1}^k w(\rho_i) = 0.$$

Pictures 4 and 5 below illustrate this cancellation. Since neither of the 2 paths intersecting at z end at b_1 , a switch at z changes the sign, but not the multiplier, and the contributions of $\vec{\rho}$ and $\delta_z(\vec{\rho})$ cancel:

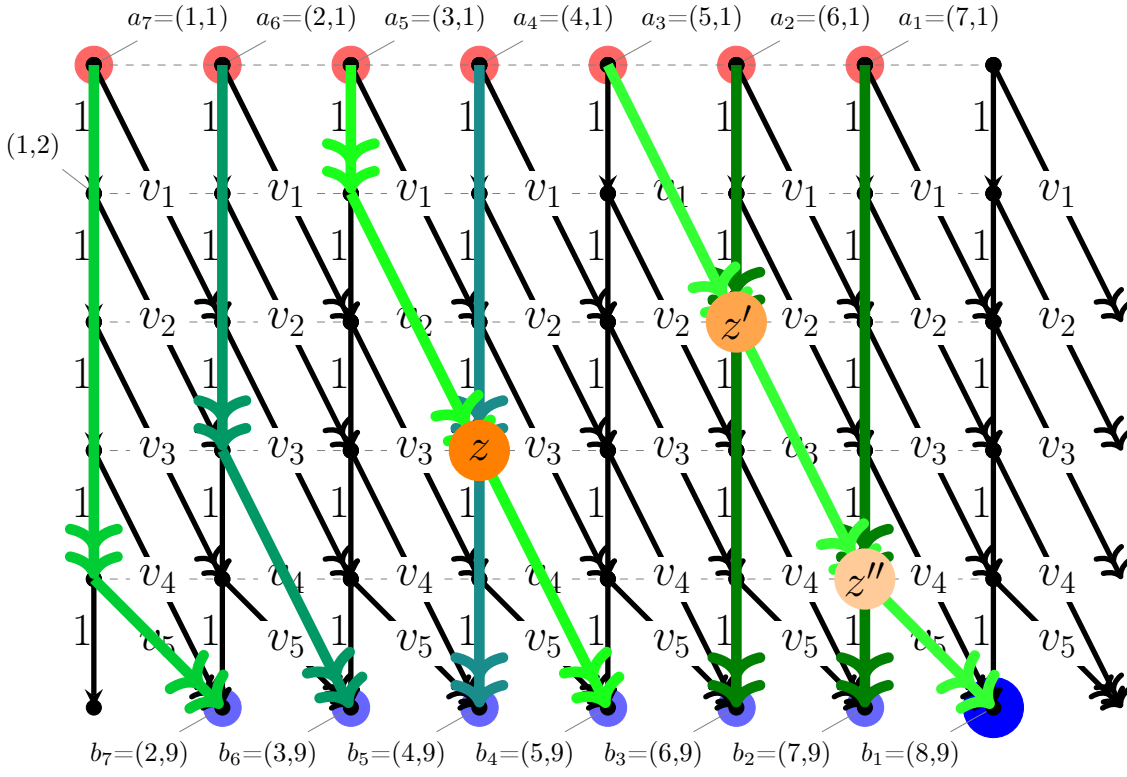


Figure 4: The grid Γ_{U_5} and multipath \vec{p} .

The path $\rho_5: a_5 \rightarrow b_4$ intersects the path $\rho_4: a_4 \rightarrow b_5$ at the point z . After the switch at z , we have the paths $\rho'_5: a_5 \rightarrow b_5$ and $\rho'_4: a_4 \rightarrow b_4$:

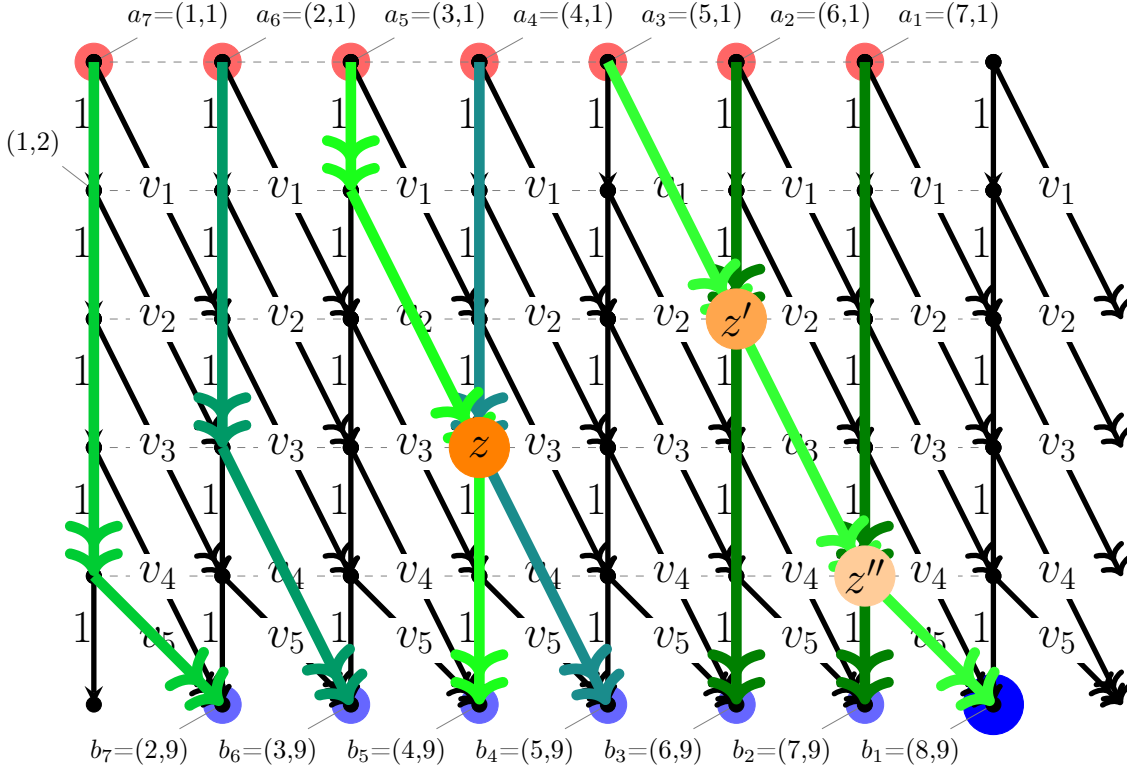


Figure 5: The grid Γ_{U_5} and multipath $\delta_z(\vec{p})$.

We denote by J_k the following set of weights vectors, which describe multipaths like on the Picture 6:

$$J_k = \{(1^{l-1}, v_{i_1} \dots v_{i_l}, v_{i_{l+1}}, \dots, v_{i_k}) \mid 1 \leq l \leq k; v_{i_j} \prec v_{i_{j+1}}, \text{ if } 1 \leq j \leq l; v_{i_1} \not\prec v_{i_{l+1}}; v_{i_j} \not\prec v_{i_{j+1}}, \text{ if } l < j \leq k\}.$$

Denote by \bar{J}_k the corresponding set of mutipaths, which are uniquely defined by the vectors of its weights.

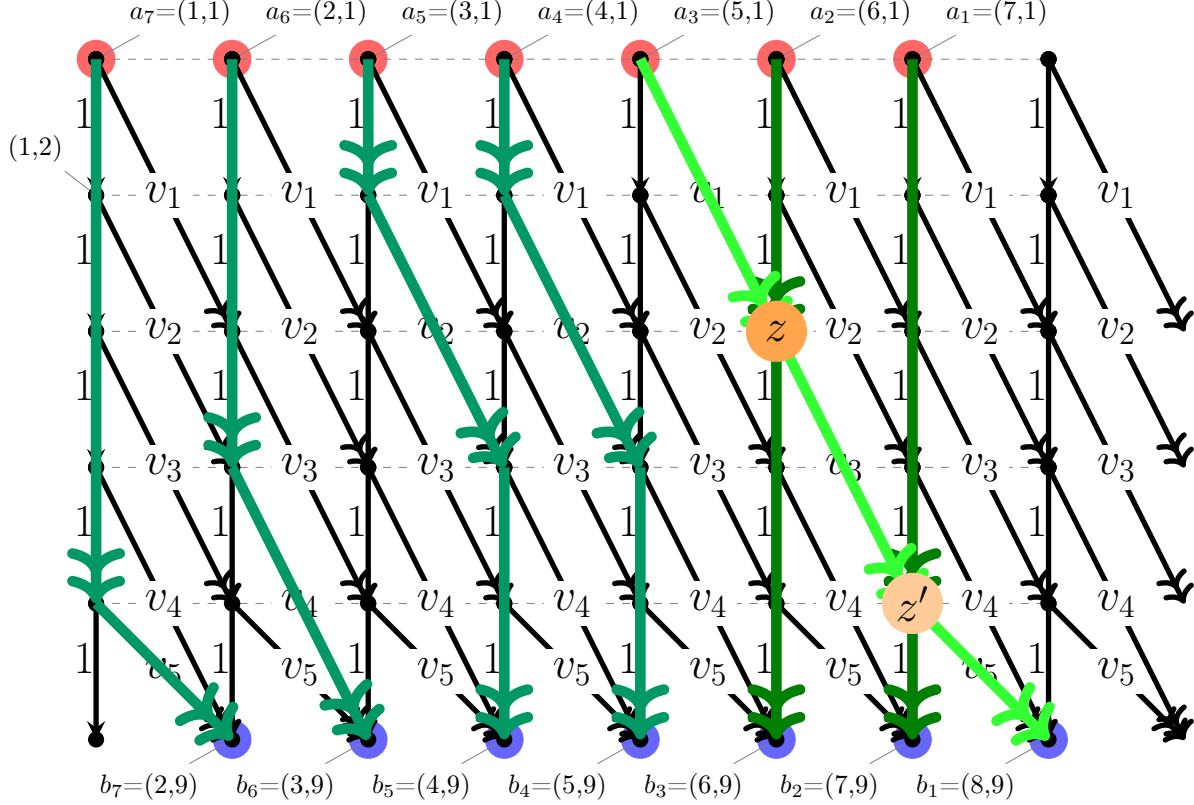


Figure 6: The grid Γ_{U_5} and a multipath without easy intersection points.

Let $\vec{\rho} \in (\bar{\Omega}_k \setminus \bar{I}_k) \setminus \bar{P}_k$.

- If $\vec{\rho}$ is intersecting, then the absolute value of the difference between the multipliers of $\vec{\rho}$ and $\delta_z(\vec{\rho})$ in the sum (13) is equal to 1:

$$|\sigma_{\vec{\rho}}^{-1}(1) - \sigma(\delta_z(\vec{\rho}))^{-1}(1)| = 1,$$

because if ρ_l goes to b_1 , then z could only be obtained as an intersection of ρ_l and ρ_{l-1} or ρ_{l+1} . Hence, since $\delta_z(\vec{\rho}) \in (\bar{\Omega}_k \setminus \bar{I}_k) \setminus \bar{P}_k$, we make a switch at z and eliminate one of the switched multipaths (from $(\bar{\Omega}_k \setminus \bar{I}_k) \setminus \bar{P}_k \setminus \bar{J}_k$) and the multiplier of the multipath with longer intersecting path (from \bar{J}_k) in the sum (13).

- If $\vec{\rho} \in (\bar{\Omega}_k \setminus \bar{I}_k) \setminus \bar{P}_k$ is non-intersecting, then its multiplier is also equal to 1. Denote the set of such multipaths by \bar{L}_k :

$$\bar{L}_k = \{\vec{\rho} \in (\bar{\Omega}_k \setminus \bar{I}_k) \setminus \bar{P}_k \mid \vec{\rho} \text{ is non-intersecting}\}.$$

Then,

$$L_k = \{(w_1, \dots, w_k) \mid w_i \not\prec w_{i+1}; \exists m, \text{ s.t. } w_m \succ \max_{1 \leq q < m} w_q\}.$$

Hence, we can rewrite the sum (13) in the following way:

$$\sum_{\vec{\rho} \in \bar{\Omega}_k \setminus \bar{I}_k \setminus \bar{P}_k} \text{sign}(\sigma_{\vec{\rho}}) \cdot \sigma_{\vec{\rho}}^{-1}(1) \prod_{i=1}^k w(\rho_i) = \sum_{\vec{\rho} \in \bar{J}_k \cup \bar{L}_k} \text{sign}(\sigma_{\vec{\rho}}) \prod_{i=1}^k w(\rho_i). \quad (14)$$

To eliminate the sum (13), we construct a sign-reversing involution on $J_k \sqcup L_k$. Let

$$A_k = \{(1^{l-1}, w_1 \cdot \dots \cdot w_l, w_{l+1}, \dots, w_k) \in J_k \mid l > 1 \text{ and } w_j \not\succ \max_{1 \leq q < j} w_q \text{ for every } 1 \leq j \leq k\}.$$

$$B_k = \{(1^{l-1}, w_1 \cdot \dots \cdot w_l, w_{l+1}, \dots, w_k) \in J_k \sqcup L_k \mid \exists m, \text{ s.t. } w_m \succ \max_{1 \leq q < m} w_q\}.$$

Then, we have

$$J_k \sqcup L_k = A_k \sqcup B_k.$$

Next, we construct a sign-reversing bijection between A_k and B_k .

First, we define map $\chi : A_k \rightarrow B_k$. If $\vec{v} = (1^{l-1}, w_1 \cdot \dots \cdot w_l, w_{l+1}, \dots, w_k) \in A_k$, then let

$$m = \max\{j \leq n \mid w_m \succ w_j \text{ for } j < m\}.$$

We set

$$\chi(\vec{v}) = (1^{l-1}, w_1 \cdot \dots \cdot w_l \cdot w_m, w_{l+1}, \dots, w_{m-1}, w_{m+1}, \dots, w_k) \in B_k.$$

Note that χ changes the sign \vec{v} by increasing its l -th weight by 1. Second, if

$$\vec{u} = (1^{l-1}, w_1 \cdot \dots \cdot w_l, w_{l+1}, \dots, w_k) \in A_k,$$

then we set

$$m' = \max\{j \leq k \mid w_l \succ w_j\},$$

and define

$$\psi(\vec{u}) = (1^{l-1}, w_1 \cdot \dots \cdot w_{l-1}, w_{l+1}, \dots, w_{m'}, w_l, w_{m'+1}, \dots, w_k) \in B_k.$$

Note that $\psi = \chi^{-1}$. For instance, the multipath from Picture 6, which belongs to A_k , is transformed to the below mutipath (Picture 7) under the action of ψ , and vice versa Picture 6 can be obtained from Picture 7 applying direct map χ :

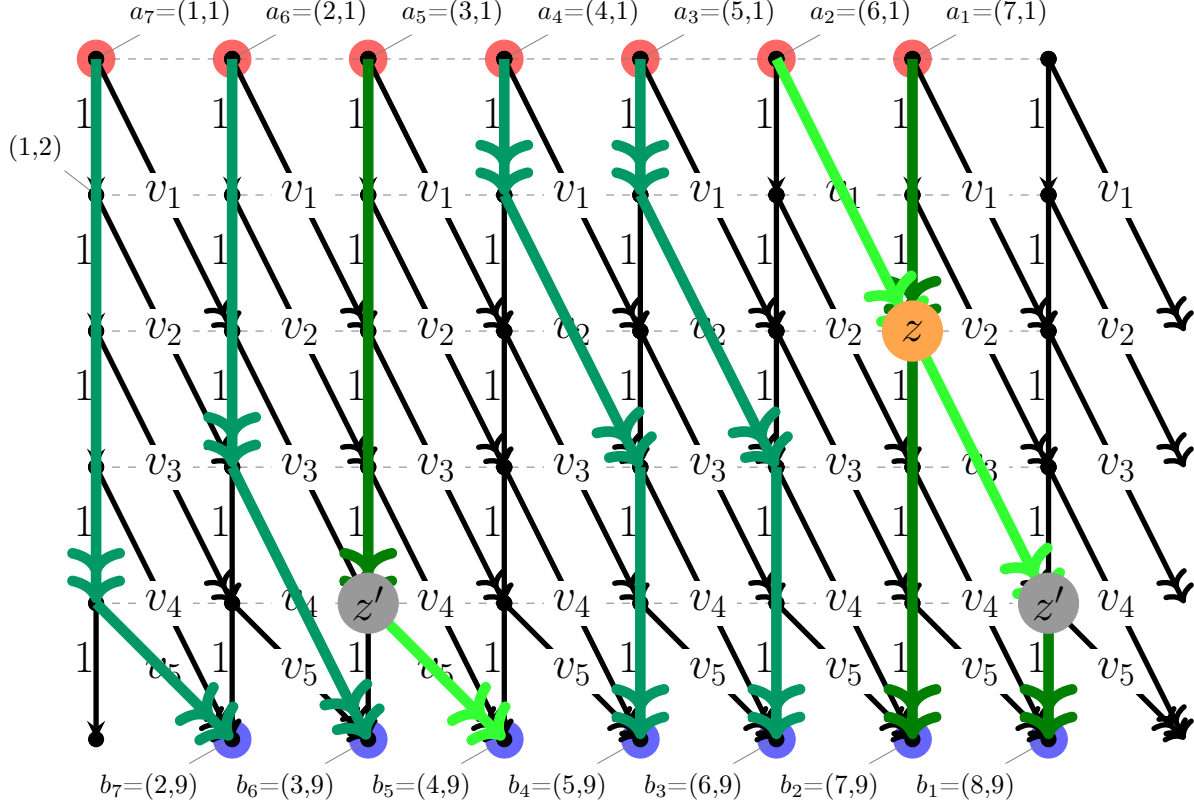


Figure 7: Image of the multipath from Picture 6 under the action of χ .

Hence, among the sums (13), (12) and (11), only the latter is non-zero, we have only the set of corrects left:

$$p_k^U = \sum_{\substack{(\rho_1, \dots, \rho_k): A \rightarrow B \\ (w(\rho_1), \dots, w(\rho_k)) \in P_k^U}} \prod_{i=1}^n w(\rho_i).$$

□

This result will play a major role in our future work. The construction of a correct sequence allows to work with m^U functions by expanding them in terms of p^U functions. For instance, using the following relation

$$m_{l,1}^U = p_l^U \cdot p_1^U - p_{l+1}^U,$$

it is easy to prove the following

Theorem 3.7. *Let*

$$M_{l,1}^U = \{(\vec{w} \mid z) \in P_l^U \times P_1^U \mid z \succ \vec{w} \vee z \prec w_l\},$$

then

$$m_{l,1}^U = \sum_{(\vec{w}; z) \in M_{l,1}^U} w_1 \cdot \dots \cdot w_l \cdot z.$$

Remark 3.8. *According to Remark 3.3, this implies $c_{n-1,1}(U) \geq 0$.*

Here, we omit the proof. This approach will be used in the next article, where positivity of some e -coefficients for $(3+1)$ -free posets will be demonstrated.

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